Centralized systemic risk control in the interbank system: weak formulation and Gamma-convergence

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Joint work with T. Li and X. Yu

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Talk Outline

Talk Outline

Mean Field Model I

- Fouque and Sun (2013): Systemic risk illustrated. *Handbook on Systemic Risk, J.P. Fouque and J.A. Langsam Eds, Cambridge University Press*.

$$dX_t^i = \frac{a}{N}\sum_{j=1}^{N}(X_t^j - X_t^i)dt + \sigma dW_t^i, \qquad i = 1, \ldots, N.$$

- Bo and Capponi (2015): Systemic risk in interbanking systems. *SIAM J. Finan. Math.* 6, 386-424.
- Biagini et al. (2019): Financial asset bubbles in banking networks. *SIAM J. Finan. Math.* 10(2), 430-465.
- Capponi et al. (2020): A dynamic system model of interbank lending-systemic risk and liquidity provisioning. *Math. Oper. Res.* 45(3), 1127-1152.

Controlled Mean Field Model I

- Carmona et al. (2015): Mean field games and systemic risk. *Commun. Math. Sci.* 13(4), 911-933.

$$dX_t^i = \frac{a}{N}\sum_{j=1}^{N}(X_t^j - X_t^i)dt + \theta_t^i dt + \sigma dW_t^i + \sigma_0 dW_t^0, \qquad i = 1, \ldots, N.$$

- Aim to minimize

$$J^i\left(\theta^1, \cdots, \theta^N\right) = \mathbb{E}\left[\int_0^T f_i\left(X_t, \theta_t^i\right)dt + g_i\left(X_T^i\right)\right],$$

where

$$f_i\left(x, \theta^i\right) = \left(\theta^i\right)^2 + \frac{q}{2}\left(\bar{x} - x^i\right)^2, \quad g_i(x) = \frac{c}{2}\left(\bar{x} - x^i\right)^2$$

Talk Outline

Financial Model I

- The log-monetary reserve of bank i satisfies

$$dX_t^{\theta,i} = \frac{a_i}{N}\sum_{j=1}^{N}(X_t^{\theta,j} - X_t^{\theta,i})dt + u_i\theta_t dt + \sigma_i dW_t^i + \sigma_0 dW_t^0, \quad t \in (0,T].$$

1. Type vector: $\xi^i := (a_i, u_i, \sigma_i)^\top \in \mathcal{O} := \mathbb{R}_+^3$.
2. Control rate implemented by the central bank: θ_t.
3. Target log-monetary reserve level determined by the central bank: Y^i.

Financial Model II

- Objective functional

$$J_N(\theta) = \mathbb{E}\left[L_N(\mathbf{X}_T^{\theta,N}, \mathbf{Y}^N) + \int_0^T R_N(\mathbf{X}_t^{\theta,N}, \mathbf{Y}^N; \theta_t)dt\right],$$

where $\mathbf{X}_t^{\theta,N} := (X_t^{\theta,1}, \ldots, X_t^{\theta,N})^\top$, $\mathbf{Y}^N := (Y^1, \ldots, Y^N)^\top$.

- Loss function

$$L_N(\mathbf{x}, \mathbf{y}) := \frac{\alpha}{N}\sum_{i=1}^N L(x_i, y_i), \quad R_N(\mathbf{x}, \mathbf{y}; \theta) := \frac{\beta}{N}\sum_{i=1}^N L(x_i, y_i) + \lambda\theta^2,$$

where $L(x_i, y_i) := |x_i - y_i|^2$.

Financial Model III

- Control problem under strong formulation:

$$\mathbb{U}^{\mathbb{P},\mathbb{F}} := \{\theta \in \mathbb{H}^2;\ \theta_t(\omega) \in \Theta \text{ a.s. on } [0,T] \times \Omega\},$$

  where $\mathbb{H}^2$ is the space of all $\mathbb{F}$-adapted and real-valued processes $\theta = (\theta_t)_{t\in[0,T]}$ satisfying $\mathbb{E}[\int_0^T |\theta_t|^2 dt] < \infty$.
- Assumption **(A$_{s1}$)**: There exists a global constant K such that $|\xi_i| \leq K$ for all $i \geq 1$. The sequence of initial log-monetary reserves $\{X_0^i\}_{i\in\mathbb{N}}$ satisfies $\sup_{i\in\mathbb{N}} \mathbb{E}[|X_0^i|^{2+\varrho}] < \infty$ for some $\varrho > 0$.
- Assumption **(A$_\Theta$)**: The policy space of $\Theta \subset \mathbb{R}$ is a (nonempty) compact and convex set.

Optimal Control with Finite Banks I

- Rewrite the system in a compact form:

$$d\mathbf{X}_t^{\theta} = \mathbf{b}(\mathbf{X}_t^{\theta}, \theta_t)dt + \Sigma d\mathbf{W}_t^0, \quad t \in [0, T],$$

- The drift term is defined by:

$$\begin{aligned}\mathbf{b}(\mathbf{X}_t^{\theta}, \theta_t) :=& \frac{1}{N}\begin{bmatrix} (1-N)a_1 & a_1 & \cdots & a_1 \\ a_2 & (1-N)a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_N & a_N & \cdots & (1-N)a_N \end{bmatrix}\begin{bmatrix} X_t^{\theta,1} \\ X_t^{\theta,2} \\ \vdots \\ X_t^{\theta,N} \end{bmatrix} + \theta_t \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^N \end{bmatrix} \\ =:& A\mathbf{X}_t^{\theta} + \theta_t \mathbf{u},\end{aligned}$$

Optimal Control with Finite Banks II

- The volatility matrix is given by

$$\Sigma := \begin{bmatrix} \sigma_0 & \sigma_1 & 0 & \cdots & 0 \\ \sigma_0 & 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_0 & 0 & 0 & \cdots & \sigma_N \end{bmatrix}_{N\times(N+1)}.$$

- Define the parameterized Hamiltonian:

$$\mathcal{H}(t,\mathbf{x},\mathbf{p},M;\mathbf{y}) := \inf_{\theta\in\Theta}\left\{\mathbf{b}(\mathbf{x},\theta)^\top\mathbf{p} + \frac{1}{2}\mathrm{tr}(\Sigma\Sigma^\top M) + R_N(\mathbf{x},\mathbf{y};\theta)\right\}.$$

Optimal Control with Finite Banks III

- Consider the following parameterized HJB equation:

$$\begin{cases} -\dfrac{\partial V}{\partial t}(t,\mathbf{x};\mathbf{y}) - \mathcal{H}(t,\mathbf{x},\nabla_{\mathbf{x}}V(t,\mathbf{x};\mathbf{y}),\nabla_{\mathbf{x}}^2 V(t,\mathbf{x};\mathbf{y});\mathbf{y}) = 0, \\ V(T,\mathbf{x};\mathbf{y}) = L_N(\mathbf{x},\mathbf{y}), \end{cases}$$

- The HJB equation admits a unique classical solution (**Theorem IV.6.2** of Fleming and Rishel (1975)).

Optimal Control with Finite Banks IV

- Optimal control:

$$\theta_t^{*,N} = f^*(t, \mathbf{X}_t^*; \mathbf{Y}) = \Pi_\Theta\left(-\frac{1}{2\lambda}\sum_{j=1}^{N} u_j \frac{\partial V(t, \mathbf{X}_t^*; \mathbf{Y})}{\partial x_j}\right),$$

where

$$\mathbf{X}_t^* = \mathbf{X}_0 + \int_0^t \mathbf{b}(\mathbf{X}_s^*, f^*(s, \mathbf{X}_s^*; \mathbf{Y}))ds + \Sigma \mathbf{W}_t, \quad t \in [0, T].$$

Our Goals

1. The convergence of optimal controls $\theta^{*,N}$ as $N \to \infty$;
2. The limit of $\theta^{*,N}$ as $N \to \infty$ is the minimizer of a so-called limiting control problem.

Challenges:

1. $\theta^{*,N}$ heavily depends on the dimension N which is coming from $\mathbf{x}, \mathbf{y}$ and V.
2. Build the rigorous connection between the problem with N banks and the mean field control problem.

Control Problem under Weak Formulation I

- Canonical space representation:

$$\Omega_\infty := \Xi^{\mathbb{N}} \times \mathcal{C}_T \times \mathcal{C}_T^{\mathbb{N}} \times \mathcal{L}_T^2, \quad \mathcal{F}_\infty := \mathcal{B}(\Omega_\infty),$$

  where $\Xi := \mathbb{R}^2$.
- coordinate process $\widehat{\mathcal{X}} := (\widehat{\boldsymbol{\zeta}}, (\widehat{W}^0, \widehat{\mathbf{W}}), \widehat{\theta})$, i.e., $\widehat{\mathcal{X}}(\omega) = \omega$ for all $\omega \in \Omega_\infty$. Here, $\widehat{\boldsymbol{\zeta}} := (\hat{\zeta}^1, \hat{\zeta}^2, \ldots)^\top$ and $\widehat{\mathbf{W}} := (\widehat{W}^1, \widehat{W}^2, \ldots)^\top$.
- Complete natural filtration $\widehat{\mathbb{F}} = (\mathcal{F}_t^{\widehat{\mathcal{X}}})_{t\in[0,T]}$ generated by the coordinate process $\widehat{\mathcal{X}}$.

Control Problem under Weak Formulation II

- The space Ω_∞ is equipped with the metric: for $(\gamma, w, \varsigma, \rho)$ and $(\hat{\gamma}, \hat{w}, \hat{\varsigma}, \hat{\rho}) \in \Omega_\infty$,

$$d((\gamma,\varsigma,w,\kappa),(\hat{\gamma},\hat{\varsigma},\hat{w},\hat{\kappa})) := d_1(\gamma,\hat{\gamma}) + d_2(\varsigma,\hat{\varsigma}) + d_3(w,\hat{w}) + d_4(\kappa,\hat{\kappa}),$$

- The metrics d_i for $i = 1, 2, 3, 4$ are given as follows:

$$d_1(\gamma,\hat{\gamma}) := \sum_{i=1}^{\infty} 2^{-i} \frac{|\gamma_i - \hat{\gamma}_i|}{1 + |\gamma_i - \hat{\gamma}_i|}, \quad \gamma = (\gamma_i)_{i=1}^{\infty},\ \hat{\gamma} = (\hat{\gamma}_i)_{i=1}^{\infty} \in \Xi^{\mathbb{N}};$$

$$d_2(\varsigma,\hat{\varsigma}) := \|\varsigma - \hat{\varsigma}\|_T = \sup_{t\in[0,T]} |\varsigma_t - \hat{\varsigma}_t|, \quad \varsigma, \hat{\varsigma} \in \mathcal{C}_T;$$

$$d_3(w,\hat{w}) := \sum_{i=1}^{\infty} 2^{-i} \frac{\|w_i - \hat{w}_i\|_T}{1 + \|w_i - \hat{w}_i\|_T}, \quad w = (w_i)_{i=1}^{\infty},\ \hat{w} = (\hat{w}_i)_{i=1}^{\infty} \in \mathcal{C}_T^{\mathbb{N}};$$

$$d_4(\kappa,\hat{\kappa}) := \|\kappa - \hat{\kappa}\|_{\mathcal{L}_T^2} = \left(\int_0^T |\kappa_t - \hat{\kappa}_t|^2\, dt\right)^{\frac{1}{2}}, \quad \kappa, \hat{\kappa} \in \mathcal{L}_T^2.$$

Control Problem under Weak Formulation III

Definition 1 (Weak Controls)

Given the law $\nu \in \mathcal{P}(\Xi^{\mathbb{N}})$, let $\mathcal{Q}(\nu)$ be the set of probability measures Q on $(\Omega_\infty, \mathcal{F}_\infty)$ satisfying

- **(i)** $Q \circ \widehat{\zeta}^{-1} = \nu$;
- **(ii)** $(\widehat{W}^0, \widehat{\mathbf{W}})$ is a sequence of independent Wiener processes on $(\Omega_\infty, \widehat{\mathcal{F}}_\infty, \widehat{\mathbb{F}}, Q)$;
- **(iii)** $\widehat{\theta} \in \mathbb{U}^{Q,\widehat{\mathbb{F}}}$.

- Control problem under weak formulation (for fixed N):

$$V_N^R(\nu) = \inf_{Q \in \mathcal{Q}(\nu)} J_N^R(Q);$$

$$J_N^R(Q) := \mathbb{E}^Q \left[L_N(\widehat{\mathbf{X}}_T^{\widehat{\theta},N}, \widehat{\mathbf{Y}}^N) + \int_0^T R_N(\widehat{\mathbf{X}}_t^{\widehat{\theta},N}, \widehat{\mathbf{Y}}^N; \widehat{\theta}_t) dt \right].$$

Main Steps:

1. Identification of explicit limiting optimization problem with infinitely many banks (i.e., $N \to \infty$).
2. Equivalence of the value functions under strong and weak formulations.
3. The minimizer of finite-dimensional optimization problem tends to the minimizer of explicit limiting optimization problem (i.e., Gamma- convergence).
4. The minimizer of the limiting optimization problem is an approximate optimal weak control to the strong control problem.

Talk Outline

Convergence of Empirical process I

- Let $Q_N \in \mathcal{Q}(\nu)$ and $\widehat{\mathcal{X}}^N$ be the corresponding coordinate process to Q_N. $\widehat{\mathbf{X}}^N := (\widehat{X}_t^{N,1}, \ldots, \widehat{X}_t^{N,N})^\top_{t\in[0,T]}$ solves the following system:

$$d\widehat{X}_t^{N,i} = \frac{a_i}{N}\sum_{j=1}^{N}(\widehat{X}_t^{N,j} - \widehat{X}_t^{N,i})dt + u_i\widehat{\theta}_t^N dt + \sigma_i d\widehat{W}_t^{N,i} + \sigma_0 d\widehat{W}_t^{N,0}.$$

- The empirical measure-valued process $\mu^N = (\mu_t^N)_{t\in[0,T]}$ under $Q_N \in \mathcal{Q}(\nu)$ is defined by

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{(\xi_i, \widehat{Y}^{N,i}, \widehat{X}_t^{N,i})} \in \mathcal{P}_2(E), \qquad t \in [0,T].$$

Convergence of Empirical process II

- Assumption **($\mathbf{A}_{s2}$)**: For any $\gamma = (x^i, y^i)_{i\geq 1} \in \Xi^{\mathbb{N}}$, define $I_N : \gamma \mapsto \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_i, y^i, x^i)} \in \mathcal{P}_2(E)$ for $N \geq 1$, there exists a measurable mapping $I_* : \Xi^{\mathbb{N}} \to \mathcal{P}_2(E)$ such that

$$\nu\left(\left\{\gamma \in \Xi^{\mathbb{N}} : \lim_{N\to\infty} \mathcal{W}_{E,2}(I_N(\gamma), I_*(\gamma)) = 0\right\}\right) = 1.$$

- Let $\mathbb{Q}^N$ be the law of empirical distribution:

$$\mathbb{Q}^N = Q_N \circ (\mu_0^N, \widehat{\theta}^N, \mu^N)^{-1}.$$

Convergence of Empirical process III

Theorem 2

Let $(\mathbf{A}_{s1})$ *holds. We assume in addition that*

$$Q_N \circ (\mu_0^N, \widehat{\theta}^N)^{-1} \Rightarrow \nu_0,$$

as $N \to \infty$, *for some* $\nu_0 \in \mathcal{P}(\mathcal{P}_2(E) \times \mathcal{L}^2)$. *Then, it holds that*

- $(\mathbb{Q}^N)_{N=1}^{\infty}$ *is relatively compact in* $\mathcal{P}_2(\hat{S})$.

where $\hat{S} := \mathcal{P}_2(E) \times \mathcal{L}_T^2 \times S$ *and* $S := C([0,T];\mathcal{P}_2(E))$

Convergence of Empirical process IV

- **Sketch of Proof**
  1. Introduce the metric on $\hat{S}$ as: for $(\nu,\theta,\rho),\ (\hat{\nu},\hat{\theta},\hat{\rho})\in\hat{S}$,

$$d_{\hat{S}}((\nu,\theta,\rho),(\hat{\nu},\hat{\theta},\hat{\rho})) := \mathcal{W}_{E,2}(\nu,\hat{\nu}) + \|\theta-\hat{\theta}\|_{\mathcal{L}_T^2} + d_S(\rho,\hat{\rho}),$$

     where $d_S(\rho,\hat{\rho}) := \sup_{t\in[0,T]}\mathcal{W}_{E,2}(\rho_t,\hat{\rho}_t)$, for $\rho,\hat{\rho}\in S$.
  2. By Villani (2003), $(\mathbb{Q}^N)_{N=1}^{\infty}$ is relatively compact in $\mathcal{P}_2(\hat{S})$ if and only if
     - (i) $(\mathbb{Q}^N)_{N=1}^{\infty}$ is tight (relatively compact) in $\mathcal{P}(\hat{S})$;
     - (ii) $\lim_{R\to\infty}\sup_{N\geq 1}\int_{\{\mu\in\hat{S};\ d_{\hat{S}}^2(\mu,\hat{\mu})\geq R\}} d_{\hat{S}}^2(\mu,\hat{\mu})\mathbb{Q}^N(d\mu)=0.$

Convergence of Empirical process V

- Characterize the limiting process: if the law of an $\hat{S}$-valued r.v. $(\tilde{\mu}_0, \tilde{\theta}, \tilde{\mu})$ defined on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ is the limiting point, then $\tilde{\mu} = (\tilde{\mu}_t)_{t\in[0,T]}$ satisfies the stochastic FPK equation that

$$\langle \tilde{\mu}_t, \phi \rangle = \langle \tilde{\mu}_0, \phi \rangle + \int_0^t \langle \tilde{\mu}_s, \mathcal{A}^{\tilde{\mu}_s, \tilde{\theta}_s} \phi \rangle ds + \sigma_0 \int_0^t \langle \tilde{\mu}_s, \phi' \rangle d\widetilde{W}_s^0, \quad \forall\, \phi \in C_b^2(\mathbb{R}),$$

where

$$\mathcal{A}^{m,\theta} \phi(x) := g^{m,\theta}(x)\phi'(x) + \frac{\sigma^2 + \sigma_0^2}{2} \phi''(x), \quad m \in \mathcal{P}_2(E),\ \theta \in \Theta,$$

and $g^{m,\theta}(x) = a\left(\int_E z m(d\xi, dy, dz) - x\right) + u\theta$.

Convergence of Empirical process VI

Proposition 3

If $\tilde{\mu}_0$ has a square-integrable density w.r.t. Lebesgue measure, then the stochastic FPK equation admits a unique solution. Thus, $\{\mathbb{Q}^N\}_{N\geq 1}$ converges in $\mathcal{P}_2(\hat{S})$.

- **Sketch of Proof**
- Well-posedness of the following linear SDE: for any fixed $\nu \in S$, $\forall\ \phi \in C_b^2(\mathbb{R})$,

$$\langle \vartheta_t, \phi\rangle = \langle \tilde{\mu}_0, \phi\rangle + \int_0^t \langle \vartheta_s, \mathcal{A}^{\nu_s, \tilde{\theta}_s}\phi\rangle ds + \sigma_0 \int_0^t \langle \vartheta_s, \phi'\rangle d\widetilde{W}_s^0. \quad (4.1)$$

Convergence of Empirical process VII

1. Change the $\mathcal{P}(E)$-valued process to an L^2-valued process:

$$T_\delta \vartheta_t(x) := \int_{\mathbb{R}^5} G_\delta(x-z)\vartheta_t(da, du, d\sigma, dy, dz),$$

   where $G_\delta(x) = \frac{1}{\sqrt{2\pi\delta}} e^{-\frac{x^2}{2\delta}}$ is the heat kernel.

2. If $\tilde{\mu}_0$ has an L^2-density with respect to Lebesgue measure., then so does ϑ_t, for any $t \in [0, T]$.

Convergence of Empirical process VIII

3. If ϑ^1 and ϑ^2 are two $\mathcal{P}(E)$-valued solutions of (4.1), then $\vartheta_t := \vartheta_t^1 - \vartheta_t^2$ satisfies

$$\widetilde{\mathbb{E}}\left[\|T_\delta \vartheta_t\|_{L^2}^2\right] \leq C \int_0^t \widetilde{\mathbb{E}}\left[\|T_\delta(|\vartheta_s|)\|_{L^2}^2\right] ds,$$

where the constant $C > 0$ is independent of ϑ^1 and ϑ^2.

4. Denote $\|\vartheta\|_{L^2}$ the L^2-norm of the density function of the signed measure ϑ.

Convergence of Empirical process IX

5. For a complete orthonormal basis $(\psi_j)_{j\geq 1}$, it holds that

$$\widetilde{\mathbb{E}}\left[\|\vartheta_t\|_{L^2}^2\right] = \widetilde{\mathbb{E}}\left[\sum_j \langle \psi_j, \vartheta_t\rangle^2\right] = \widetilde{\mathbb{E}}\left[\sum_j \lim_{\delta\to 0}\langle T_\delta \psi_j, \vartheta_t\rangle^2\right]$$
$$= \widetilde{\mathbb{E}}\left[\sum_j \lim_{\delta\to 0}\langle T_\delta \vartheta_t, \psi_j\rangle_{L^2}^2\right] \leq \underline{\lim}_{\delta\to 0} C\int_0^t \widetilde{\mathbb{E}}\left[\|T_\delta(|\vartheta_s|)\|_{L^2}^2\right] ds$$
$$\leq C\int_0^t \widetilde{\mathbb{E}}\left[\|\,|\vartheta_s|\,\|_{L^2}^2\right] ds = C\int_0^t \widetilde{\mathbb{E}}\left[\|\vartheta_s\|_{L^2}^2\right] ds.$$

6. By Gronwall's inequality, the linear SDE (4.1) has a unique solution.

Convergence of Empirical process X

- Well-posedness of the following conditional Mckean-Vlasov equation:

$$\begin{cases} dX_t = \tilde{a}\left(X_t - \int_E x\,\nu_t(d\xi, dy, dx)\right)dt + \tilde{u}\tilde{\theta}_t dt + \tilde{\sigma}d\widetilde{W}_t + \sigma_0 d\widetilde{W}_t^0, \\ \nu_t = \mathcal{L}((\tilde{a}, \tilde{u}, \tilde{\sigma}, \widetilde{Y}, X_t)|\widetilde{\mathcal{G}}_t), \end{cases}$$

where $\tilde{\zeta} = (\tilde{a}, \tilde{u}, \tilde{\sigma}, \widetilde{Y}, X_0)$ is an E-valued r.v. with $\widetilde{\mathbb{P}} \circ \tilde{\zeta}^{-1} = \tilde{\mu}_0$ and $\widetilde{\mathcal{G}}_t := \sigma(\tilde{\zeta}) \vee \sigma(\widetilde{W}_s^0, \tilde{\theta}_s;\ s \leq t)$.

1. For any $r > 0$, define Banach space

$$\mathbb{H}_r := \left\{X:\ \widetilde{\mathbb{F}}\text{-adapted process},\ \|X\|_r := \widetilde{\mathbb{E}}\left[\int_0^T e^{-rt}|X_t|dt\right] < \infty\right\}.$$

Convergence of Empirical process XI

2. For any $X^{(i)} \in \mathbb{H}_r$ and $\nu_t^{(i)} = \mathcal{L}((\tilde{a}, \tilde{u}, \tilde{\sigma}, \widetilde{Y}, X_t^{(i)})|\widetilde{\mathcal{G}}_t)$, $i = 1, 2$, define

$$\mathcal{Z}_t(X^{(i)}) := X_0 + \tilde{a} \int_0^t \left(\int_E x \, \nu_s^{(i)}(d\xi, dy, dx) - X_s^{(i)} \right) ds + \tilde{u} \int_0^t \tilde{\theta}_s ds + \tilde{\sigma} \widetilde{W}_t + \sigma_0 \widetilde{W}_t^0.$$

3. For any $r > 0$,

$$\left\| \mathcal{Z}(X^{(1)}) - \mathcal{Z}(X^{(2)}) \right\|_r \leq \frac{2K}{r} \left\| X^{(1)} - X^{(2)} \right\|_r .$$

4. Then the result follows from the contraction mapping theorem.

Convergence of Empirical process XII

- Well-posedness of the stochastic FPK equation.
  1. Let $\nu^{(1)}, \nu^{(2)}$ be two solutions. Consider the following SDE:

$$dX_t^{(1)} = \tilde{a}\left(X_t^{(1)} - \int_E x\,\nu_t^{(1)}(d\xi, dy, dx)\right) dt + \tilde{u}\tilde{\theta}_t dt + \tilde{\sigma} d\widetilde{W}_t + \sigma_0 d\widetilde{W}_t^0.$$

  2. Then $\vartheta_t^{(1)} = \mathcal{L}((\tilde{a}, \tilde{u}, \tilde{\sigma}, \widetilde{Y}, X_t^{(1)})|\widetilde{\mathcal{G}}_t)$ is a solution to the linear equation (by Itô's formula) [$\nu^{(1)}$ solves FPK]

$$\langle \vartheta_t^{(1)}, \phi\rangle = \langle \tilde{\mu}_0, \phi\rangle + \int_0^t \langle \vartheta_s^{(1)}, \mathcal{A}^{\nu_s^{(1)}, \tilde{\theta}_s}\phi\rangle ds + \sigma_0 \int_0^t \langle \vartheta_s^{(1)}, \phi'\rangle d\widetilde{W}_s^0.$$

  3. Thus $\nu^{(1)} = \vartheta^{(1)}$, and then $(X^{(1)}, \nu^{(1)})$ is a solution to the conditional Mckean-Vlasov equation.

Convergence of Empirical process XIII

4. Similarly, $(X^{(2)}, \nu^{(2)})$ also is a solution to the conditional Mckean-Vlasov equation.
5. By the uniqueness of solution to the conditional Mckean-Vlasov equation, we have $\nu^{(1)} = \nu^{(2)}$.

Convergence of Objective Functionals

Theorem 4

Let $(\mathbf{A}_{s1})$ *and* $(\mathbf{A}_{s2})$ *hold. Then,* $\lim_{N\to\infty} J_N^R(Q) = J^R(Q)$. *Here,*

$$J^R(Q) := \alpha \int_S \langle \rho_T, L\rangle \widehat{\mathbb{Q}}_\mu(d\rho) + \beta \int_0^T \left(\int_S \langle \rho_t, L\rangle \widehat{\mathbb{Q}}_\mu(d\rho) \right) dt$$
$$+ \lambda \mathbb{E}^Q \left[\int_0^T |\widehat{\theta}_t|^2 dt \right],$$

where $\widehat{\mathbb{Q}}_\mu$ *is the weak limit of*

$$\widehat{\mathbb{Q}}_\mu^N := Q_N \circ (\hat{\mu}^N)^{-1}.$$

- **Sketch of Proof**

1. By Thm 7.12 in Villani (2003),

$$h_N(t) := \int_S \langle \rho_t, L\rangle \widehat{\mathbb{Q}}_\mu^N(d\rho) \to \int_S \langle \rho_t, L\rangle \widehat{\mathbb{Q}}_\mu(d\rho), \quad \forall\, t \in [0,T], \quad N \to \infty.$$

2. By the uniform integrability of $\{h_N(t)\}_{N\geq 1}$, we have

$$\lim_{N\to\infty} \int_0^T h_N(t)dt = \int_0^T \left(\int_S \langle \rho_t, L\rangle \widehat{\mathbb{Q}}_\mu(d\rho) \right) dt.$$

Talk Outline

Gamma-Convergence I

- Metrize $\mathcal{Q}(\nu)$ by taking 2nd-order Wasserstein distance $\mathcal{W}_2$.
- Γ-convergence of $(J_N^R(Q))_{N=1}^{\infty}$ on metric space $(\mathcal{Q}(\nu), \mathcal{W}_2)$ is defined as (see, e.g. Braides (2014)):

Definition 5 (Gamma-Convergence)

We call $J_N^R : \mathcal{Q}(\nu) \to \mathbb{R}$ Γ-converges to $J^R : \mathcal{Q}(\nu) \to \mathbb{R}$, i.e., small$J^R = \Gamma\text{-}\lim_{N\to\infty} J_N^R$ on $\mathcal{Q}(\nu)$, if the following conditions hold:

(i) (lim inf inequality): For any $Q \in \mathcal{Q}(\nu)$ and every sequence $(Q_N)_{N=1}^{\infty}$ converging to Q in $(\mathcal{Q}(\nu), \mathcal{W}_2)$, we have that $\liminf_{N\to\infty} J_N^R(Q_N) \geq J^R(Q)$;

(ii) (lim sup inequality): For any $Q \in \mathcal{Q}(\nu)$, there exists a sequence $(\hat{Q}_N)_{N=1}^{\infty}$ which converges to Q in $(\mathcal{Q}(\nu), \mathcal{W}_2)$ (this sequence is said to be a Γ- realising sequence), we have that $\limsup_{N\to\infty} J_N^R(\hat{Q}_N) \leq J^R(Q)$.

Gamma-Convergence II

- The following proposition implies both the $\liminf$ and $\limsup$ inequalities.

Proposition 6

Let assumptions $(\mathbf{A}_{s1})$, $(\mathbf{A}_{s2})$ *and* $(\mathbf{A}_{\Theta})$ *hold. For any* $\{Q_N\}_{N\geq 1}, Q \subset \mathcal{Q}(\nu)$ *satisfying* $\lim_{N\to\infty} \mathcal{W}_{\Omega_\infty,2}(Q_N, Q) = 0$, *let* $(\widehat{\zeta}^N, (\widehat{W}^{N,0}, \widehat{\mathbf{W}}^N), \widehat{\theta}^N)$ *(resp.* $(\widehat{\zeta}, (\widehat{W}^0, \widehat{\mathbf{W}}), \widehat{\theta})$*) be the corresponding coordinate process to* Q_N *(resp.* Q*). If* $I_*(\widehat{\zeta})$ *has a square-integrable density (under* Q*) w.r.t. Lebesgue measure, then we have*

$$\lim_{N\to\infty} J_N^R(Q_N) = J^R(Q).$$

Precompactness of the Minimizer I

Construct a precompact sequence of minimizers:

- The equivalence of the value functions under strong and weak formulations. That is,

$$\inf_{\theta \in \mathbb{U}^{\mathbb{P},\mathbb{F}}} J_N(\theta) = \inf_{Q \in \mathcal{Q}(\nu)} J_N^R(Q).$$

- Continuity of the objective functional. $J_N(\theta) : \mathbb{U}^{\mathbb{P},\mathbb{F}} \to \mathbb{R}$ is continuous with respect to the metric induced by the $\mathbb{H}^2$-norm.
- By Ekeland's variational principle: there exists a minimizing sequence $\{\theta^k\}_{k\geq 1} \subset \mathbb{U}^{\mathbb{P},\mathbb{F}}$, s.t.

$$J_N(\theta^k) \leq J_N(\theta) + \frac{1}{k}\|\theta^k - \theta\|_{\mathbb{H}^2}, \qquad \forall\, \theta \in \mathbb{U}^{\mathbb{P},\mathbb{F}}.$$

Precompactness of the Minimizer II

- Characterize the minimizing sequence. There exists $\chi^k \in \mathbb{H}^2$ with $\|\chi^k\|_{\mathbb{H}^2} \leq 1$, such that

$$\theta_t^k = \Pi_\Theta\left(-\frac{1}{2\lambda}\sum_{i=1}^N u_i p_t^{k,i} - \frac{1}{2\lambda k}\chi_t^k\right).$$

Here $(\mathbf{p}^k, \mathbf{q}^k)$ is the unique solution to the adjoint equation

$$d\mathbf{p}_t^k = -\left[A^\top \mathbf{p}_t^k + \frac{2\beta}{N}(\mathbf{X}_t^k - \mathbf{Y})\right]dt + \mathbf{q}_t^k d\mathbf{W}_t, \quad t \in [0, T),$$
$$\mathbf{p}_T^k = \nabla_{\mathbf{x}} L_N(\mathbf{X}_T^k, \mathbf{Y}) = \frac{2\alpha}{N}(\mathbf{X}_T^k - \mathbf{Y}),$$

where $X^k = (X_t^k)_{t\in[0,T]}$ satisfies $d\mathbf{X}_t^k = \mathbf{b}(\mathbf{X}_t^k, \theta_t^k)dt + \Sigma d\mathbf{W}_t^0$.

Precompactness of the Minimizer III

- Construct admissible relaxed control sequence: $Q^k := \mathbb{P}\circ(\zeta,(W^0,\mathbf{W}),\theta^k)^{-1}$ and show the tightness of $(Q^k)_{k\geq 1}$. Thus, Q^k converge to some $Q^{N,*}$ weakly (along a sub-sequence).
- Using Skorokhod's representation theorem, we have:

$$J_N^R(Q^{N,*}) = \lim_{k\to\infty} J_N(\theta^k) = \inf_{\theta\in\mathbb{U}^{\mathbb{P},\mathbb{F}}} J_N(\theta) = \inf_{Q\in\mathcal{Q}(\nu)} J_N^R(Q).$$

- The sequence of minimizers $(Q^{N,*})_{N\geq 1}$ is tight.

Main Results

- The main implication of (i) Γ-convergence and (ii) the precompactness of the sequence of minimizers:

Theorem 7

Let **(A_{s1})** *and* **(A_{s2})** *hold. Then, as* $N \to \infty$,

$$\inf_{Q \in \mathcal{Q}(\nu)} J_N^R(Q) \to \min_{Q \in \mathcal{Q}(\nu)} J^R(Q),$$

where the minimum of $J^R(Q)$ *exists. Moreover, if the minimizer* $(Q^{N,*})_{N=1}^{\infty} \subset \mathcal{Q}(\nu)$ *(up to a subsequence) converges to some* $Q^* \in \mathcal{Q}(\nu)$ *(the existence of* Q^* *has been guaranteed), then* Q^* *minimizes* $J^R(Q)$.

Approximate Optimal Weak Control

Corollary 8

Let $Q^* \in \mathcal{Q}(\nu)$ *be the* **minimizer** *of* $J^R(Q)$. *Then*

$$\lim_{N\to\infty}\left|J_N^R(Q^*) - \inf_{\theta\in\mathbb{U}^{\mathbb{P},\mathbb{F}}} J_N(\theta)\right| = 0.$$

- In fact, we have that, as $N \to \infty$,

$$\left|J_N^R(Q^*) - \inf_{\theta\in\mathbb{U}^{\mathbb{P},\mathbb{F}}} J_N(\theta)\right| = \left|J_N^R(Q^*) - \inf_{Q\in\mathcal{Q}(\nu)} J_N^R(Q)\right|$$
$$\leq \left|J_N^R(Q^*) - J^R(Q^*)\right| + \left|\inf_{Q\in\mathcal{Q}(\nu)} J^R(Q) - \inf_{Q\in\mathcal{Q}(\nu)} J_N^R(Q)\right| \to 0.$$

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Thank you!